Math 308N **Midterm 1** **Winter 2018**

Your Name

Your Signature

Student ID #

Honor Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8.5'' \times 11''$ sheet of handwritten notes (both sides OK). Do not share notes. No photocopied materials are allowed.
- Calculators are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
Spicy Bonus	6	
Total	50	

1. (20 points) Indicate whether the given statement is true or false (2 pts) and give justification as to why it is true or false(2 pts).

a) [4 pts] Let $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ be a linearly independent set in $\mathbb{R}^3$, then the set $\{c\vec{\mathbf{u}}_1, c\vec{\mathbf{u}}_2, c\vec{\mathbf{u}}_3\}$ is linearly independent, for all scalars $c \neq 0$.

Algebraic solution: TRUE. If $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ is linearly independent then the equation

$$x_1\vec{\mathbf{u}}_1 + x_2\vec{\mathbf{u}}_2 + x_3\vec{\mathbf{u}}_3 = \vec{\mathbf{0}}$$

only has the trivial solution. For $c \neq 0$, we have that

$$cx_1\vec{\mathbf{u}}_1 + cx_2\vec{\mathbf{u}}_2 + cx_3\vec{\mathbf{u}}_3 = c(x_1\vec{\mathbf{u}}_1 + x_2\vec{\mathbf{u}}_2 + x_3\vec{\mathbf{u}}_3) = \vec{\mathbf{0}}$$

This implies that any solution to the second equation is also a solution to the first. Since the original equation only as the trivial solution, so does the second, hence $\{c\vec{\mathbf{u}}_1, c\vec{\mathbf{u}}_2, c\vec{\mathbf{u}}_3\}$ is linearly independent.

Geometric solution: TRUE. If we are given three linearly independent vectors in $\mathbb{R}^3$, then any one of them cannot lie in the plane spanned by the other two. Since multiplication by c only scales the vectors in question and does not change their direction, scaling the linearly independent vectors doesn't effect their geometric relationship to one another, hence they remain linearly independent.

b) [4 pts] There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 3 \quad \text{and} \quad T\left(\begin{bmatrix}4\\0\\4\end{bmatrix}\right) = 9$$

FALSE. If T were linear it would have to satisfy the second property of linear transformations, namely that for any nonzero scalar r, $T(r\vec{\mathbf{u}}) = rT(\vec{\mathbf{u}})$. This means that if $T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 3$, then

$$T\left(\begin{bmatrix}4\\0\\4\end{bmatrix}\right) = T\left(4\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 4T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = 4\cdot(3) = 12$$

c) [4 pts] If $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$, and $\vec{\mathbf{v}}$ are vectors in $\mathbb{R}^3$, and $\vec{\mathbf{v}}$ is in span$\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$, then $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{v}}\}$ can never span $\mathbb{R}^3$.

TRUE. If $\vec{\mathbf{v}}$ is in span$\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$, then $span\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\} = span\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{v}}\}$. If $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{v}}\}$ were to span $\mathbb{R}^3$ then $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ would span $\mathbb{R}^3$, but no two vectors can span $\mathbb{R}^3$, hence $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{v}}\}$ can never span $\mathbb{R}^3$.

Give an example of each of the following. If it is not possible write "NOT POSSIBLE". No justification is needed if it is not possible.

d) [2 pt] A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$, which reflects the unit square about the x-axis. (Note: Take the unit square to lie in the first quadrant. Giving the matrix of T, if it exists, is a sufficient answer).

> The simplest linear transformation that reflects the unit square about the x- axis, is the one that sends $\mathbf{e}_1$ to $\mathbf{e}_1$ ($T(\mathbf{e}_1) = \mathbf{e}_1$), and $\mathbf{e}_2$ to $-\mathbf{e}_2$ ($T(\mathbf{e}_2) = -\mathbf{e}_2$). Calling this linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, we know that it's associated matrix is given by $A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)]$ hence
>
> $$T(\vec{x}) = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

e) [2 pt] A set of 4 vectors in $\mathbb{R}^3$ that spans $\mathbb{R}^3$ and is linearly **dependent**.

> There are many examples. A simple one is
>
> $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

f) [2 pt] 3 linearly independent vectors $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ satisfying the equation $2\vec{\mathbf{u}}_1 + 3\vec{\mathbf{u}}_2 - 4\vec{\mathbf{u}}_3 = \vec{\mathbf{0}}$.

> NOT POSSIBLE. By the definition of linear independence, the equation $x_1\vec{\mathbf{u}}_1 + x_2\vec{\mathbf{u}}_2 + x_3\vec{\mathbf{u}}_3 = \vec{\mathbf{0}}$ only has the trivial solution, so no linearly independent vectors can satisfy the relation given, else it would be a non-trivial solution.

g) [2 pts] A homogeneous linear system with strictly more variables than equations, having exactly one solution.

> NOT POSSIBLE. The system must have infinitely many solutions (See theory review chapter 1 question 1c). If there are more variables than equations, the matrix corresponding to this system will have more columns than rows. If we consider these columns as vectors, then we will be considering a set of n vectors in $\mathbb{R}^m$, where $n > m$. Since the system is homogeneous, it has exactly one or infinitely many solutions, but it has exactly one solution (the trivial solution) precisely when these column vectors are linearly independent. Since there are more vectors than the dimension of the space, these vectors must be linearly dependent, hence we must have infinitely may solutions.

2. (10 points) Consider the following linear system with a and b nonzero constants.

$$\begin{cases} \quad x_1 - 3x_2 + \ \ x_3 = 4 \\ \ 2x_1 \qquad\quad - 8x_3 = -2 \\ -6x_1 + 6x_2 + ax_3 = b \end{cases}$$

a) [5 pts] For what values of a and b does the system have infinitely many solutions?

We begin by row reducing the corresponding augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & 0 & -8 & -2 \\ -6 & 6 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 6 & -10 & -10 \\ 0 & -12 & a+6 & b+24 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 6 & -10 & -10 \\ 0 & 0 & a-14 & b+4 \end{array}\right]$$

This system has infinitely many solutions precisely when the last row is $0 = 0$ since this gives way to the existence of free variables. In order for this to happen we must have $a - 14 = b + 4 = 0$ hence the values we need are $a = 14$ and $b = -4$.

a) [5 pts] Give an example of a and b where the system has exactly one solution.

The system has exactly one solution when the reduced matrix has a pivot in every column. In other words, when $a - 14 \neq 0$. Note that in this case b can be anything. The easiest case is $a = 15$ and $b = -3$ but there are many other examples.

3. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + 3x_2 + 2x_3 \\ 4x_1 - 12x_2 - 8x_3 \end{bmatrix}$.

a) [3pts] Determine the matrix associated to T. That is, find the matrix A such that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

Recall that the matrix associated to T is given by $A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ T(\mathbf{e}_3)]$. Computing these values we see that

$$T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ -12 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$$

Thus the associated matrix is given by $A = \begin{bmatrix} -1 & 3 & 2 \\ 4 & -12 & -8 \end{bmatrix}$.

b) [3 pts] Is T one-to one? Explain your answer.

For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, if $m > n$, then T is never one-to-one. Since $m = 3$ and $n = 2$, T is not one-to-one.

c) [4pts] Is T onto? If not, find a vector not in the range of T.

Recall that T is onto if and only if the columns of A span $\mathbb{R}^2$. If T is not onto, then augmenting A with an arbitrary vector will allow us to find a vector that is not in the range of T. Augmenting and row reducing we have that

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & a \\ 4 & -12 & -8 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & a \\ 0 & 0 & 0 & 4a+b \end{array}\right]$$

so we can see that if $4a + b \neq 0$ then the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ satisfying this condition will not be in the span of the columns of A. Since the span of the columns of A is precisely the range of T, any vector satisfying this condition will not be in the range. An easy example is taking $a = 1, b = -3$, hence $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is not in the range of T.

4. (10 points) a) [4 pts] Let $\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}$. Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbb{R}^3$.

The big theorem implies that any set of 3 vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$ if and only if the set is linearly independent, so we will show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent since it is easier to compute. Augmenting with the zero vector we have that

$$\left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\1&2&2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&1&2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&0&1&0\end{array}\right]$$

We can see the corresponding system has only the trivial solution, hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, and by the big theorem, spans $\mathbb{R}^3$.

b) [4 pts] Express the vector $\mathbf{v} = \begin{bmatrix}3\\5\\11\end{bmatrix}$ as a linear combination of the vectors from the previous part.

We apply the same row reductions as above, except with the matrix now augmented with $\mathbf{v}$.

$$\left[\begin{array}{ccc|c}1&1&0&3\\0&1&1&5\\1&2&2&11\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&3\\0&1&1&5\\0&1&2&8\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&3\\0&1&1&5\\0&0&1&3\end{array}\right]$$

This puts the system in triangular form and we can now apply back substitution. We have that $x_3 = 1$ and since $x_2 + x_3 = 5$, $x_2 = 2$. Looking at the last equation, we see that $x_1 + x_2 = 3 \implies x_1 = 1$, hence $(x_1, x_2, x_3) = (1, 2, 3)$ which implies that

$$\begin{bmatrix}3\\5\\11\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\2\end{bmatrix} + 3\begin{bmatrix}0\\1\\2\end{bmatrix} = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$$

.

c) [2 pts] Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{u}_1) = \begin{bmatrix}1\\3\end{bmatrix}, T(\mathbf{u}_2) = \begin{bmatrix}2\\0\end{bmatrix}, T(\mathbf{u}_3) = \begin{bmatrix}1\\-1\end{bmatrix}$$

Compute $T(\mathbf{v})$ (where $\mathbf{v}$ is the same vector as in part b).

Recalling that linearity of T implies that $T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v})$, we can apply the same definition to compute $T(\mathbf{v})$.

$$T(\mathbf{v}) = T(\mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3) = T(\mathbf{u}_1) + 2T(\mathbf{u}_2) + 3T(\mathbf{u}_3) = \begin{bmatrix}1\\3\end{bmatrix} + 2\begin{bmatrix}2\\0\end{bmatrix} + 3\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}8\\0\end{bmatrix}$$

Super Spicy Bonus Question

5. (6 points) a) [4pts] Let $f(x) = ax^2 + bx + c$ denote an arbitrary polynomial of degree 2, with constants a, b, and c. To each such polynomial, associate the vector

$$ax^2 + bx + c \leftrightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Let T be the linear transformation given by $T(f(x)) = f'(x)$. In other words, T is the linear transformation that eats a degree 2 polynomial and spits out its derivative (Yes it is linear!). Find the matrix corresponding to T. (Hint: Remember that if we are given a linear transformation T as above, its matrix is completely determined by $T(\mathbf{e}_i)$, i.e. $T(\mathbf{x}) = A\mathbf{x}$ where $A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ T(\mathbf{e}_3)]$

Using the hint, we set out to compute $T(\mathbf{e}_1), T(\mathbf{e}_2)$, and$T(\mathbf{e}_3)$, using the above association at each step. First, we have if the vector $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then it corresponds to the polynomial x^2 (since b and c are zero). Computing $T(\mathbf{e}_1)$ is the same as taking the derivative of the polynomial corresponding to $\mathbf{e}_1$, which gives us the polynomial $2x$. Writing this like above, we can write $2x = 0x^2 + 2x + 0$ so $a = c = 0$ and $b = 2$, hence $T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

Looking at $\mathbf{e}_2$ we associate the polynomial x, whose derivative is 1. The constant 1 is associated with the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ hence $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Continuing with this same process again we see that $\mathbf{e}_3$ corresponds to the polynomial 1, whose derivative is 0, hence corresponds to the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now, we can use the formula for the matrix of A, and we see that the derivative of any degree two polynomial is characterized by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

b) [2 pts] Is T one-to one? Is it onto? Explain your answer.

By the big theorem, we know that T is one-to-one if and only if it is onto, since the matrix for T is 3×3. Recalling that T is one-to-one if and only if the columns of A are linearly independent, we can see that one of the columns of A is the zero vector, which makes any set of vectors linearly dependent. This implies that T is neither one-to-one nor onto.